

Crystallized merons and inverted merons in the condensation of spin-1 Bose gases with spin-orbit coupling

S.-W. Su,¹ I.-K. Liu,² Y.-C. Tsai,³ W. M. Liu,⁴ and S.-C. Gou²

¹*Department of Physics, National Tsing Hua University, Hsinchu 30013, Taiwan*

²*Department of Physics, National Changhua University of Education, Changhua 50058, Taiwan*

³*Department of Photonics, Feng Chia University, Taichung 40724, Taiwan*

⁴*Institute of Physics, Chinese Science Academy, Beijing 100190, P. R. China*

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The non-equilibrium dynamics of a rapidly quenched spin-1 Bose gas with spin-orbit coupling is studied. By solving the stochastic projected Gross-Pitaevskii equation, we show that crystallization of merons can occur in a spinor condensate of ^{87}Rb . Analytic form and stability of the crystal structure are given. Likewise, inverted merons can be created in a spin-polarized spinor condensate of ^{23}Na . Our studies provide a chance to explore the fundamental properties of meron-like matter.

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Spin-orbit coupling (SOC) [1] is an ubiquitous quantum phenomenon, which links the internal (spin) and orbital (linear or angular momentum) degrees of freedom of a particle. The best known example of SOC arises in the motion of electrons in the atom, where the electron's orbiting around the nucleus can affect the orientation of electron's spin. Recently, SOC with tunable strength has been realized by Lin *et al.* [2] by shining two orthogonal laser beams intersecting in a pseudo spin-1/2 Bose-Einstein condensate (BEC) of ^{87}Rb [3]. The lasers are detuned from Raman resonance so that the momentum and spin can couple via exchanging photons in the two orthogonal beams. This opens new possibilities to simulate the role of SOC for a wide range of phenomena in condensed matter physics by using ultracold atoms, promising applications to quantum computing [4], spintronics devices [5, 6] and topological insulators [7–9]. Inspired by the experimental realization in Ref. [2], theoretical extensions using atomic BECs have been studied by a number of authors, including the direct incorporation of SOC into the spin-1/2, 1 and 2 BECs [10–13], and non-trivial ground state structures have been predicted. Novel excitations have also been predicted in fermionic gases [14], such as the Rashba pairing bound state (Rashbon) [15, 16].

So far, all studies on the ultracold atoms with SOC were focused in the cases of zero temperature. It is fundamentally important to see how the nonlocal nature of SOC affects the pattern of spontaneous symmetry breaking. In this work, we study the non-equilibrium dynamics during the condensation of a spin-1 Bose gas with SOC. In particular, we focus on the formation of topological defects in the limit of rapid temperature quench. According to Kibble-Zurek mechanism [17, 18], topological defects can be created via phase transitions at finite temperatures, which are caused by spontaneous symmetry breaking and thermal fluctuations near the critical point. By solving the stochastic projected Gross-Pitaevskii equation (SPGPE), we show that, in a spin-1 BEC, the combination of SOC, spin-exchange interaction and thermal fluctuations can generate meron-like excita-

tions. A meron is a peculiar topological defect that was originally hypothesized as a half-instanton in the particle physics [19] and later a half-skyrmion in the quantum Hall systems [20, 21]. Since merons carry half unit of topological charge, it is believed that isolated merons can only be observed when particular boundary conditions are imposed. By far, merons have been created in the superfluid $^3\text{He-A}$ [22, 23] in a rotating cylinder, and the spinor BEC with a special constraining magnetic field [24]. In what follows, we show that, stable collective excitations such as the crystalline orders of merons and other isolated variants can be created in a rapidly quenched spinor BEC with SOC.

The order parameter of a spin-1 BEC is given by $\Psi = (\Psi_1, \Psi_0, \Psi_{-1})^T$, where T stands for the transpose and Ψ_{m_F} ($m_F = \pm 1, 0$) denotes the macroscopic wave function of the atoms condensed in the spin state $|1, m_F\rangle$. The total particle number, N , and total magnetization, M , are normalized by $\int |\Psi|^2 d^3r = N$, and $\int (|\Psi_1|^2 - |\Psi_{-1}|^2) d^3r = M$. In the following, we consider SOC of the form $H_{so} = \sum_{\alpha} V_{\alpha} \hat{p}_{\alpha} \hat{F}_{\alpha}$, where V_{α} , \hat{p}_{α} and \hat{F}_{α} are respectively the coupling strength, the linear momentum and the 3×3 matrix of the spin-1 angular momentum in the $\alpha (= x, y, z)$ direction. In the absence of magnetic field, the dynamics of Ψ is described by the following coupled nonlinear Schrödinger equations,

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \Psi_j &= \hat{H}_j^{GP} \Psi_j \\ &= \hat{\mathcal{H}} \Psi_j + g_s \sum_{\alpha} \sum_{n, k, l} \left(\hat{F}_{\alpha} \right)_{jn} \Psi_n \Psi_k^* \left(\hat{F}_{\alpha} \right)_{kl} \Psi_l \\ &\quad + \frac{\hbar}{i} \sum_{\alpha} \sum_n V_{\alpha} \left(\hat{F}_{\alpha} \right)_{jn} \partial_{\alpha} \Psi_n \quad (j, k, l, n = -1, 0, 1) \end{aligned} \quad (1)$$

Here $\hat{\mathcal{H}} = -\hbar^2 \nabla^2 / 2m + U(\mathbf{r}) + g_n |\Psi|^2$ denotes the spin-independent part of the Hamiltonian, with $U(\mathbf{r})$ being the trapping potential. The coupling constants g_n and g_s characterizing the density-density and spin-exchange interactions, respectively, are related to the s -wave scatter-

ing lengths a_0 and a_2 in the total spin channels $F_{total} = 0$, 2 by $g_n = 4\pi\hbar^2(a_0 + 2a_2)/3m$, $g_s = 4\pi\hbar^2(a_2 - a_0)/3m$ [25, 26]. Note that $g_n > 0$, whereas g_s can be either positive or negative. A spin-1 BEC is said to be ferromagnetic (FM) when $g_s < 0$, and antiferromagnetic (AFM) when $g_s > 0$. As we shall focus on the dynamics of spin texture, we introduce the basis set Ψ_α ($\alpha = x, y, z$), such that $\Psi_{\pm 1} = (\pm\Psi_x + i\Psi_y)/\sqrt{2}$ and $\Psi_0 = \Psi_z$. As a result, $\hat{F}_\alpha|\alpha\rangle = 0$, and the spin texture, which is parallel to the local magnetic moment, is defined by $\mathbf{S}(\mathbf{r}) = i\tilde{\Psi}^\dagger \times \tilde{\Psi}/|\Psi|^2$ where $\tilde{\Psi} = (\Psi_x, \Psi_y, \Psi_z)^T$ [27]. For later use, we define the unit vector $\mathbf{s}(\mathbf{r}) = \mathbf{S}(\mathbf{r})/|\mathbf{S}(\mathbf{r})|$.

In the mean-field theory approximation, the dynamics

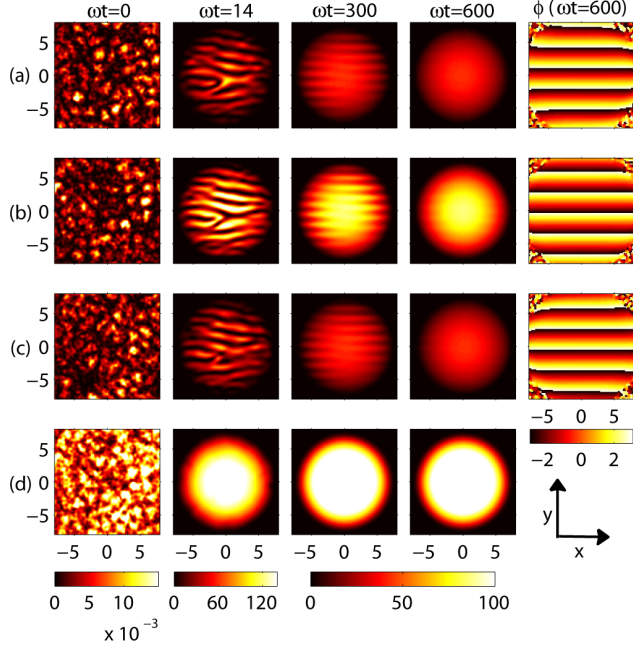


FIG. 1: (color online) Snapshots of (a) $|\Psi_{-1}|^2$, (b) $|\Psi_0|^2$, (c) $|\Psi_1|^2$, and (d) $|\Psi|^2$ of an ^{87}Rb (FM) spinor BEC during quench with $V_x = 1$, $V_y = 2$. The rightmost column shows the phase profile of equilibrium state. The particle numbers in the equilibrium state are $N_{\pm 1} \approx 3.65 \times 10^3$, $N_0 = 7.32 \times 10^3$.

of a BEC at nonzero temperatures can be described by the SPGPE [28–30], based on the assumption that the system can be treated as a condensate band in contact with a thermal reservoir comprising of all non-condensed particles. The condensate band is described by the truncated Wigner method [31] including the projected c -field method, while the non-condensate band is by the quantum kinetic theory [32, 33]. A direct generalization to the spinor BECs with SOC leads to the following set of coupled SPGPEs [34]

$$\frac{\partial \Psi_j}{\partial t} = \mathcal{P} \left\{ -\frac{i}{\hbar} \hat{H}_j^{GP} \Psi_j + \frac{\gamma_j}{k_B T} \left(\mu - \hat{H}_j^{GP} \right) \Psi_j + \frac{dW_j}{dt} \right\} \quad (2)$$

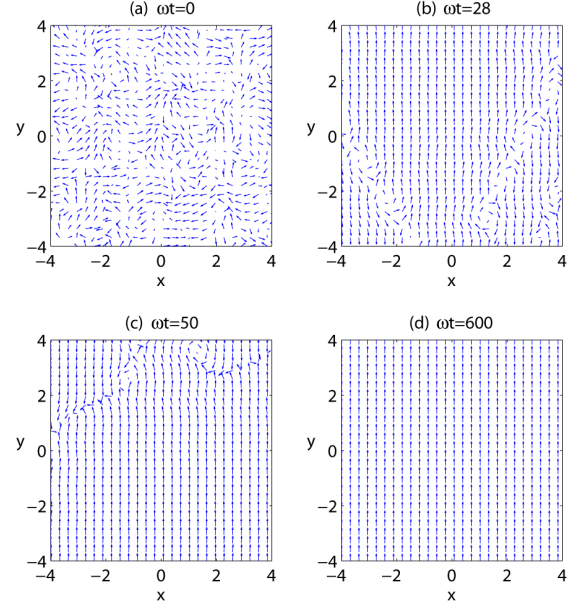


FIG. 2: Snapshots of the vector field $\mathbf{s}(\mathbf{r})$. Arrays of merons of the same handedness can be seen in (b). All merons drift away towards the periphery of the condensate as shown in (c) and (d).

where T and μ denote the final temperature and chemical potential, γ_j the growth rate for the j -th component, and dW_j/dt is the complex-valued white noise associated with the condensate growth. The projection operator \mathcal{P} restricts the dynamics of the spinor BEC in the lower energy region below the cutoff energy E_R . For simplicity, we shall consider a 2-dimensional isotropic trap, $U(\mathbf{r}) = m\omega^2(x^2 + y^2)/2$. The numerical procedures for solving SPGPEs, are described as follows. First, the initial state of each Ψ_j is sampled by using the grand-canonical ensemble for free ideal Bose gas at a temperature T_0 below the critical temperature and of chemical potentials $\mu_{j,0}$. The spatial dependence of the initial state can be specified as a linear combination of plane waves with discretized momentum $\mathbf{k} = 2\pi(n_x, n_y)/L$ ($n_x, n_y \in Z$ and L is the size of the system), i.e., $\Psi_j(t=0) = \sum_{\mathbf{k}}^{E_R} a_{j,\mathbf{k}} \psi_{\mathbf{k}}(\mathbf{r})$, where $\psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}}$. The condensate band lies below the energy cutoff $E_R > E_{\mathbf{k}} = \hbar^2 |\mathbf{k}|^2 / 2m$. The distribution is sampled by $a_{j,\mathbf{k}} = (N_{j,\mathbf{k}} + 1/2)^{1/2} \eta_{j,\mathbf{k}}$ where $N_{j,\mathbf{k}} = [\exp((E_{j,\mathbf{k}} - \mu_{j,0})/k_B T_0) - 1]^{-1}$ and $\eta_{j,\mathbf{k}}$ are the complex Gaussian random variables with moments $\langle \eta_{j,\mathbf{k}} \eta_{j,\mathbf{k}'} \rangle = \langle \eta_{j,\mathbf{k}}^* \eta_{j,\mathbf{k}'}^* \rangle = 0$ and $\langle \eta_{j,\mathbf{k}} \eta_{j,\mathbf{k}'}^* \rangle = \delta_{\mathbf{k}\mathbf{k}'}$. Second, to simulate the thermal quench, the temperature and chemical potential of the non-condensate band are altered to the new values $T < T_0$ and $\mu > \mu_{j,0}$. For convenience, we adopt the oscillator units in the numer-

ical computations, and the length, time and energy are respectively scaled in units of $\sqrt{\hbar/m\omega}$, ω^{-1} and $\hbar\omega$.

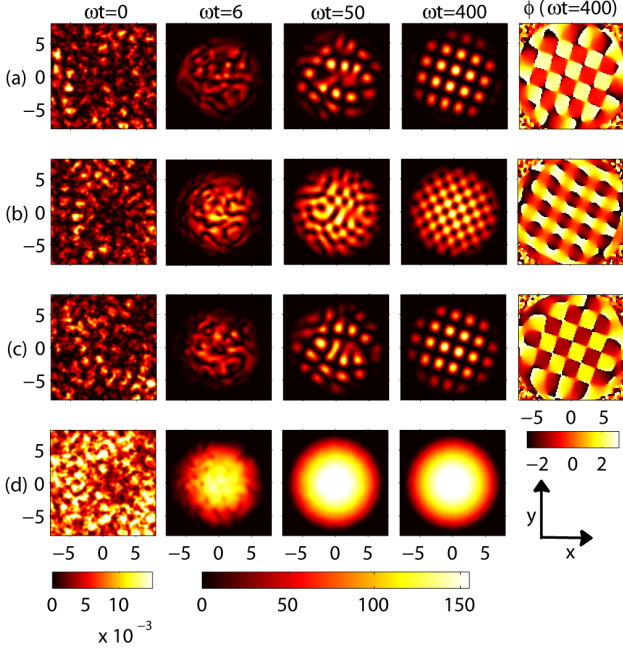


FIG. 3: (color online) Snapshots of (a) $|\Psi_{-1}|^2$, (b) $|\Psi_0|^2$, (c) $|\Psi_1|^2$, and (d) $|\Psi|^2$ of the ^{87}Rb (FM) spinor BEC at zero magnetization with $V = 1.8$. The rightmost column shows the phase profile of equilibrium state. The corresponding particle numbers are $N_{\pm 1} \approx 3.56 \times 10^3$, $N_0 = 7.08 \times 10^3$.

We first study the condensation of ^{87}Rb ($g_s < 0$) with SOC. The initial states are sampled at $k_B T_0 = 500$, $\mu_{j,0} = 2$. Here we set $E_R = 50.5$, $k_B T = 0.067$, $\mu = 25$, and $\hbar\gamma_j/k_B T = 0.05$. In this paper, we consider the in-plane coupling, i.e., $V_x, V_y \neq 0$ and $V_z = 0$. We begin with the cases of $|V_x| \neq |V_y|$. In Fig. 1, stripe structure develops in the density profile of each component during condensation, but fade away gradually. On the other hand, the equilibrium phase profiles become periodically striped along one direction, which is exactly the plane wave (PW) state found in Ref. [11]. The evolution of the spin texture is shown in Fig. 2. Clearly, spin domains form shortly after the quench starts. The domain walls comprise of sheets of spin vortex which drift outward and disappear while the system approaches equilibrium marked by a uniform spin alignment on the xy plane. The PW state is bound to occur when $|V_x| \neq |V_y|$, or $|V_x| = |V_y| = V < 0.8$. When $|V_x| = |V_y| = V > 0.8$, periodic structures, which are caused by the formation of grids of dark soliton in $\Psi_{\pm 1}$ and vortex lattice in Ψ_0 , may appear in the equilibrium density profiles of all components in Fig. 3. In Fig. 4(a)-4(b), the spin texture consists of two interlacing square lattices of spin vortex with opposite vorticity, and we denote it as the spin vortex lattice (SVL) state. When $V > 0.8$, PW and SVL state appear alternatively with definite probabilities depend-

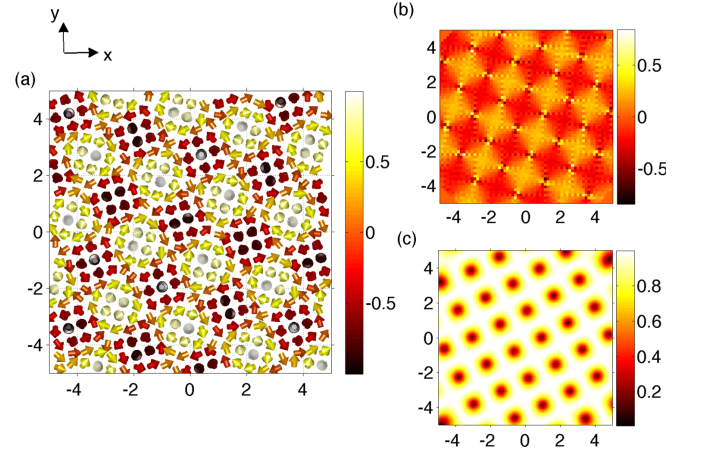


FIG. 4: (Color online) (a) The 3-dimensional orientation of the spin textures in Fig. 1. The color indicate the magnitude of s_z . (b) The topological charge density σ of the equilibrium spin texture. (c) The corresponding spatial distribution of $|\mathbf{S}(\mathbf{r})|$.

ing on the magnitude of V . Using the imaginary time propagation method, the SVL state is shown to have a higher energy than the PW state. In the SVL state, all spin vortices have similar structure — the central spin always orients to the z axis, while the others increasingly twist and finally lie on the xy plane, forming a circulation pattern away from the center. The topological charge density, $\sigma = \mathbf{s} \cdot (\partial_x \mathbf{s} \times \partial_y \mathbf{s}) / 4\pi$, is plotted in Fig. 4(b) and the vortex with left/right-handed circulation has a positive/negative σ . This is just the Mermin-Ho vortex, or *meron* [35]. Integrating $Q = \int \sigma d^2r$ over a primitive unit cell, we identify $Q = \pm 1/2$, corresponding to merons and antimerons, respectively. Due to the FM nature of the condensate, a meron and an antimeron will pair up to form a vortex dipole, which was predicted in the bilayer quantum Hall system as the lowest energy excitation [21] and in a fast-rotating highly spin-polarized spinor BEC [27] and now acts as the building block of the SVL. For conciseness, in the following, we shall not distinguish the merons from antimerons unless specially noted.

To gain more insight into the SVL state, we plot the equilibrium momentum wave functions, $\tilde{\Psi}_i(\mathbf{p})$, and find that they are all sharply peaked at $\mathbf{p} \approx \pm 2.8\mathbf{e}_x, \pm 2.8\mathbf{e}_y$. This suggests that the equilibrium state might be consisted of 4 plane waves with $\mathbf{p} = \pm \mathbf{q}, \pm \mathbf{q}'$, where $\mathbf{q} \cdot \mathbf{q}' = 0$ and $|\mathbf{q}| = |\mathbf{q}'|$. The counter-propagating modes with $\mathbf{p} = \pm \mathbf{q}, \pm \mathbf{q}'$ form two orthogonal standing waves that superimpose to generate the periodic structures in all density profiles in Fig. 3. In the absence of trapping potential, it was shown that, the one-particle Hamiltonian, $p^2/2m + V\mathbf{p} \cdot \mathbf{F}$, is minimized by $|\mathbf{p}| = V$ [11, 12],

and hence

$$\begin{aligned}\tilde{\Psi}_{\pm 1}(\mathbf{p}) &= \frac{\sqrt{N}e^{\mp i\theta}}{4} \left[-\delta\left(\mathbf{p} - \frac{V\mathbf{q}}{|\mathbf{q}|}\right) - \delta\left(\mathbf{p} + \frac{V\mathbf{q}}{|\mathbf{q}|}\right) \right. \\ &\quad \left. \mp \delta\left(\mathbf{p} - \frac{V\mathbf{q}'}{|\mathbf{q}'|}\right) \mp \delta\left(\mathbf{p} + \frac{V\mathbf{q}'}{|\mathbf{q}'|}\right) \right], \\ \tilde{\Psi}_0(\mathbf{p}) &= \sqrt{\frac{N}{8}} \left[\delta\left(\mathbf{p} - \frac{V\mathbf{q}}{|\mathbf{q}|}\right) - \delta\left(\mathbf{p} + \frac{V\mathbf{q}}{|\mathbf{q}|}\right) \right. \\ &\quad \left. + i\delta\left(\mathbf{p} - \frac{V\mathbf{q}'}{|\mathbf{q}'|}\right) - i\delta\left(\mathbf{p} + \frac{V\mathbf{q}'}{|\mathbf{q}'|}\right) \right],\end{aligned}\quad (4)$$

where $\theta = \tan^{-1}(q_y/q_x)$. Consequently, the spin textures are given by,

$$\mathbf{S}(\mathbf{r}) = (\cos\theta \cos u \sin v + \sin\theta \sin u \cos v, \cos\theta \sin u \cos v - \sin\theta \cos u \sin v, \cos u \cos v) \quad (5)$$

where $u = V(x \cos\theta + y \sin\theta)$ and $v = V(-x \sin\theta + y \cos\theta)$. The above analytical results fairly reproduce the spin texture plotted in Fig. 4(a) with an effective V in the presence of trapping potential. From Fig. 4(c), we see that the spin density vanishes at the center of a vortex quadruple (4 mutually adjoining merons), which happens to be the density troughs of $\Psi_{\pm 1}$ that is filled by the particles of Ψ_0 . The meron and polar cores are centered at $\mathbf{r}_{\text{meron/polar}} = R(\theta) \mathbf{d}_{\text{meron/polar}}$ where $R(\theta)$ is the rotation matrix on xy plane, $\mathbf{d}_{\text{meron}} = (n, l)\pi/V$, $\mathbf{d}_{\text{polar}} = (n + 1/2, l + 1/2)\pi/V$, and $n, l \in Z$. It should be noted that the cores of the meron are located at the vortices of Ψ_0 . With Eqs. (3)-(4), the energies of PW and SVL states for a homogeneous spinor BEC can be calculated analytically. As PW and SVL states both yield the same minimized energy for the single particle Hamiltonian with a fixed N , we only need to consider the interacting energies. It is straightforward to show that both PW and SVL states lead to the identical density-density interaction energies and thus the only difference is the spin-exchange term. As a result, the spin-exchange interaction energies per unit area, $\epsilon_{spin} = g_s N \int S^2(\mathbf{r}) d^2r$, are $g_s N$ and $3g_s N/4$ for PW and SVL states, respectively. In the current case, $g_s < 0$, and thus the SVL state has a higher energy.

The SVL state is sustained by the vorticity originated mostly from SOC. As μ rather than N is fixed in our simulations, increasing V will increase N and the mass vortices in Ψ_0 . Since PW and SVL states are gapped by an amount of energy proportion to N , it is expected that there exists a threshold V_c , beyond which the SVL will barely appear. This implies that the PW state totally prevails in the large V limit. The value of V_c can be estimated by considering a homogeneous spinor BEC, in which the core size of the mass vortex nucleated in the $m_F = 0$ component is $\xi_0 \sim \hbar/|\Psi_0| \sqrt{2mg_n}$. The instability of SVL state occurs when the lattice constant is comparable to ξ_0 , such that a polar core will partially overlap with the cores of contiguous merons, giving $V_c \sim \pi |\Psi_0| \sqrt{2mg_n}/\hbar$. By the same token, if the size

of the condensate is smaller than the lattice constant, the condensate does not accommodate SVL, which occurs when $V < 0.8$ in our previous result in a trap. In the limit of vanishing SOC, the spatial periodicity in the phase profiles of PW and SVL states is infinitely prolonged, leading to uniform distribution in all phase profiles — the manifestation of a normal spinor BEC. We

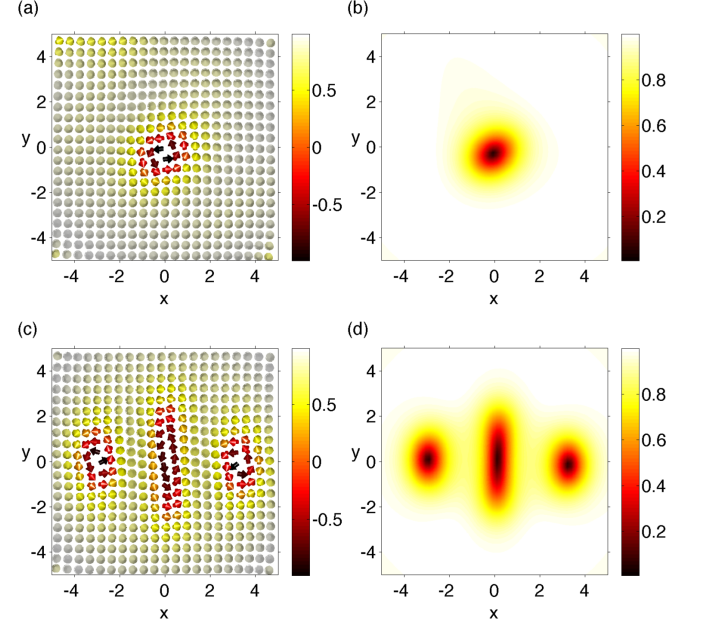


FIG. 5: (color online) Spin textures and the associated distribution of $|S(\mathbf{r})|$ in a ^{23}Na (AFM) spinor BEC for the inverted merons with (a), (b) $M/N = 0.88$; (c), (d) $M/N = 0.78$. The particle numbers in the equilibrium state are $N_1 \approx 4.66 \times 10^4$, $N_0 = 5.14 \times 10^3$, $N_{-1} \approx 3.74 \times 10^2$ in (a) and $N_1 \approx 4.18 \times 10^4$, $N_0 = 9.51 \times 10^3$, $N_{-1} \approx 9.03 \times 10^2$.

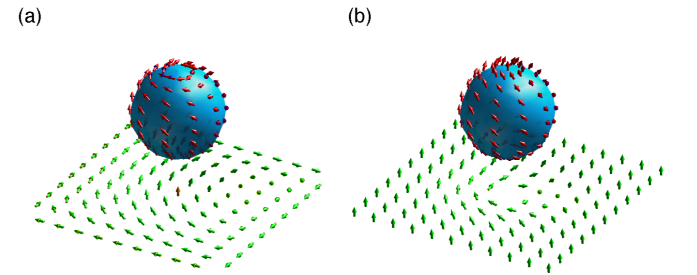


FIG. 6: (Color online) (a) The stereographic projection of the spin textures onto S^2 for (a) a meron; (b) an inverted meron.

now consider the spinor BEC of ^{23}Na ($g_s > 0$) with SOC. Using the same parameter setting, we conclude that if the spin-1 gas is unpolarized initially, no spin texture will be produced in the condensate, a conclusion already given in Ref. [11]. The equilibrium spin texture for a spin-polarized state with $M/N \approx 0.88$ is shown in Fig. 5(a),

where all spins in the outer region point to the $+z$ direction, tilting gradually, and finally lie on the xy plane while approaching the center region. Consequently, a polar core is formed at the center, with Ψ_0 filling in the vortices of both $\Psi_{\pm 1}$ which are separated but locate proximately to the center. Such a configuration corresponds to a charge of $Q = -1/2$. In Fig. 6 the stereographic projections for a meron and the texture of Fig. 5(a) are plotted. Clearly, Fig. 6(b) is exactly the inverted figure of Fig. 6(a), and thus we term the structure of Fig. 5(a) the *inverted meron*. In Fig. 5(c)-5(d), three inverted merons with distorted cores are formed with $M/N \approx 0.78$. It is easy to verify that the single isolated inverted meron has a lower energy.

In summary, we have investigated the non-equilibrium dynamics of spin-1 BECs with SOC in the limit of rapid quench. Crystallization of merons and polar core vortices are predicted to arise in the FM spinor BEC. Likewise, isolated inverted merons can be created in the highly po-

larized AFM spinor BEC. Our studies provide a method to create nontrivial structure of merons and thus an opportunity to probe into the fundamental properties of meron-like matter. Following the experimental methods in Ref.[2], our predictions can be realized in principle, except in lifting the degeneracy of the hyperfine spin states of $F = 1$, a weaker magnetic field is needed to avoid the decoupling of spin states due to quadratic Zeeman shift. The experiment may start by trapping a thermal spin-1 Bose gas and then rapidly quench it to the quantum degenerate regime. Finally, the equilibrium spin texture can be resolved *in situ* by using the polarization-dependent phase-contrast technique [36].

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